QUASI-ISOTROPIC APPROXIMATE EQUATIONS OF WEAK ACOUSTOELASTICITY

A. É. Puro

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The fundamental problem of acoustoelasticity is that of determining the relations between the parameters of a propagating ultrasonic wave and the components of the preload tensor [1-3]. Since acoustoelastic effects are very small, the problems considered are usually [3] given in a linear formulation. Most studies are confined to the motion of a wave along one of the principal directions of the prestress tensor, when the acoustoelasticity resolvents are simplified [1-5]. An exception is [6], where formulas were obtained in the first approximation for the absolute phase differences. The phenomenon of birefringence (rotation of the polarization of the shear waves in a ray and a change of phases as a result of that rotation) was not explained with sufficient accuracy there. Below, quasi-isotropic approximation [7] is used to analyze this effect. With the formulas obtained the theory developed in integrated photoelasticity can be applied, without substantial changes, to problems of acoustoelasticity [8, 9]. The topics discussed in this paper were not adequately reflected in a comparative analysis of these phenomena [4].

1. We proceed from the expression for the density of strain energy (internal energy) per unit mass of the medium [1]:

$$W=\frac{1}{\rho}\left(\frac{\lambda}{2}K_1^2+\mu K_2+\frac{1}{6}\nu_1K_1^3+\nu_2K_1K_2+\frac{4}{3}\nu_3K_3\right).$$

Here $K_1 = E_{ii}$; $K_2 = E_{ij}E_{ji}$; $K_3 = E_{ij}E_{kj}E_{ki}$; λ, μ are Lamé coefficients; ν_1, ν_2, ν_3 are third-order elastic constants: $E_{ik} = (1/2)(\partial/\partial x_k)u_i^0 + (\partial/\partial x_i)u_k^0 + (\partial/\partial x_i)u_n^0(\partial/\partial x_k)u_n^0$ is the Cauchy–Green strain tensor; $u_i^0 = u_i + w_i$ is the strain vector, represented as the sum of the strains u_i and w_i due to a preload and an ultrasonic wave; and ρ is the density of the medium in the unstrained state; the tensor summation rule is applied over recurring indices. The solution is obtained in the Lagrangian orthogonal coordinate system. The medium is assumed to be isotropic and homogeneous initially.

The resolvents are derived by using an expansion in three small parameters [6]; 1) the linearized preload tensor σ_{ij} is assumed to be small relative to the Lamé constants ($|(\sigma_{ij}|/\mu = \varepsilon_0 \le 10^{-3}$-$10^{-5}$); 2) the stress tensor of the ultrasonic wave is an order of magnitude smaller than σ_{ij}; 3) the ultrasonic wavelength λ is at least an order of magnitude smaller than the characteristic size of the preload field. It can be shown [1] that to within the first order of smallness (in ε_0) inclusively the motion of the ultrasonic wave is described by

$$\frac{\partial}{\partial x_m}\left[C_{nmjs}\frac{\partial}{\partial x_s}w_j\right]=\rho\frac{\partial^2}{\partial t^2}w_n. \tag{1.1}$$

The components of the tensor C_{nmjs}, linearized in the prestrains ε_{ij}, respectively, are

$$C_{nmjs}=c_{nmjs}+c_{mskl}\varepsilon_{kl}\delta_{nj}+c_{nmps}\frac{\partial}{\partial x_p}u_j+c_{pmjs}\frac{\partial}{\partial x_p}u_n+c_{nmjskl}\varepsilon_{kl}.$$

Here $c_{nmjs} = \lambda\delta_{nm}\delta_{js} + \mu(\delta_{nj}\delta_{ms} + \delta_{ns}\delta_{mj})$; the values of the tensor $c_{nmjsk\ell}$ are given in [1, 2].

We note Eq. (1.1) in fact describes the propagation of an ultrasonic wave in an anisotropic medium with an asymmetric stress tensor: $C_{nmjs} = C_{jsnm}$ has a lower symmetry than C_{nmjs} and $C_{nmjs} = C_{nmsj}$ does not always obtain. At the same time this equation does not take wave attenuation into account and the law of conservation of energy holds for it [10].

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To derive this law we fold both parts of (1.1) with $\partial w_n/\partial t$ and after manipulations we write

$$\frac{\partial}{\partial t} E + \frac{\partial}{\partial x_m} S_m = 0, \tag{1.2}$$

where $E = (1/2)[\rho((\partial/\partial t)w_n)((\partial/\partial t) w_n) + C_{nmjs}((\partial/\partial x_m)w_n)((\partial/\partial x_s)w_j)]$ denotes the density of the acoustic wave energy; and $S_m = -C_{nmjs}(\partial/\partial x_s)w_j(\partial/\partial t)w_n$ is the energy flux tensor.

In accordance with the method of geometric acoustics, we look for a solution of Eq. (1.1) in the form

$$w_i = W_i \exp i\omega(\varphi - t). \tag{1.3}$$

Substitution of the trial solution (1.3) transforms Eq. (1.1) (the general exponent $\exp i\omega\cdot(\varphi - t)$ has been omitted from the equation);

$$\begin{gathered}[\rho\omega^2 W_n - C_{nmjs}k_m k_s W_j] + i\left[2\rho\omega \frac{\partial}{\partial t} W_n + C_{nmjs}\left(k_m \frac{\partial}{\partial x_s} W_j + k_s \frac{\partial}{\partial x_m} W_j\right) + \right.\\ \left. + W_j \frac{\partial}{\partial x_m}(C_{nmjs}k_s)\right] + \frac{\partial}{\partial x_m}\left[C_{nmjs}\frac{\partial}{\partial x_s} W_j\right] - \rho \frac{\partial^2}{\partial t^2} W_n = 0\end{gathered} \tag{1.4}$$

($k_i = \omega\partial\varphi/\partial x_i$ are the wave vector components).

With the assumptions made (the characteristic size of the load field is considerably greater than the wavelength λ) the absolute value of the wave vector $k = 2\pi/\lambda$ appears as a large parameter and we use the ray method [5] to solve Eq. (1.4).

In the zeroth approximation of this method solving the system of differential equations reduces to solving the algebraic system

$$[\rho\omega^2\delta_{nj} - k^2 C_{n3j3}]W_j = 0. \tag{1.5}$$

Without loss of generality in our discussion we introduce a local coordinate system such that the axis $x_3 = z$ is directed along the wave vector **k**.

We make the axis $x_1 = x$ and $x_2 = y$ coincide with the directions of the acoustic tensor $c_{nj} = C_{n3j3}$ in the x, y plane. Henceforth by analogy with photoelasticity we refer to these directions as the quasiprincipal directions in the x, y plane.

The system of homogeneous equations (1.5) has a nontrivial solution if the determinant of the system is zero:

$$\det [\rho v^2\delta_{nj} - c_{nj}] = 0. \tag{1.6}$$

The condition determines the phase velocity $v = \omega/k$ of longitudinal (v_p) and transverse (v_s) waves.

Equation (1.6) can be solved only by numerical methods for an arbitrary anisotropic solid. The distinctive features of the solution of (1.6) for the linearized model of an artibrary nonlinear solid were considered in [10]. The analysis of (1.6) is simplified within the framework of the assumptions used (in practice the wave velocities are less than 10^{-4} [4]). In the coordinate system adopted the characteristic equation can be rewritten as

$$(c_{33} - \rho v^2) = \frac{c_{31}^2}{c_{11} - \rho v^2} - \frac{c_{32}^2}{c_{22} - \rho v^2}. \tag{1.7}$$

From (1.7) we see that apart from terms of the first order of smallness inclusively in ε_0 the quasilongitudinal wave velocity v_p is found in the same way as in propagation along one of the principal directions, i.e., the right side of Eq. (1.7) is assumed to be zero:

$$v_p = v_{p0} + \alpha(\sigma_{xx} + \sigma_{yy}) + \beta\sigma_{zz}. \tag{1.8}$$

Here $v_{p0} = \sqrt{(\lambda + \mu)/\rho}$ is the velocity of the longitudinal wave in the absence of stresses; and α and β are constants determined in terms of first-order and third-order elastic constants [2].

Substituting v_p into Eq. (1.5), we find the polarization of the quasilongitudinal wave (c_{13}, c_{23}, $\lambda + \mu$) (as the eigenvector of the acoustic matrix) to within the first order of smallness inclusively. The quantity W_z is determined by the initial data, the ray trajectories, and the transport equation. Since the acoustic tensor is perturbed only slightly by the prestress, the curvature of rays in the initially isotropic solid can be disregarded.

The transport equation, according to the ray method, is obtained by setting the first-order terms in k equal to zero in Eq. (1.1). Instead of the transport equation we can use the law of conservation of energy, which, to within the first-order terms, becomes

$$\frac{\partial}{\partial t} E_0 + \frac{\partial}{\partial x_m}(v_m E_0) = 0,$$

where $E_0 = (1/2)\rho\omega^2 W_z^2$ is the average value of the energy density; $v_{m3} = [\delta_{m3} + (1 - \delta_{m3})\cdot(C_{3m33} + c_{m3})/c_{33}]v_p$ is the group velocity vector of the quasilongitudinal wave. As we see from the above formula, the group velocity does not have the same direction as the phase velocity; problems associated with this effect are discussed from the measuring aspects in [11].

2. We write the zeroth approximation of quasitransverse waves in the form [7]

$$w_i = W_i \exp i\left(\int_0^z k(l)\,dl - \omega t\right) \quad (i = x, y). \tag{2.1}$$

Here $k^2 = 2\rho\omega^2/(c_{11} + c_{22})$ determines the wave number of the transverse waves. After substituting (2.1) into (1.4) with allowance for the smallness of the terms, we obtain a truncated system

$$i\left[2\rho\omega\frac{\partial}{\partial t} + 2c_{11}k\frac{\partial}{\partial z}\right]W_x = k^2\left[\frac{c_{11} - c_{22}}{2}W_x + c_{12}W_y\right],$$
$$i\left[2\rho\omega\frac{\partial}{\partial t} + 2c_{22}k\frac{\partial}{\partial z}\right]W_y = k^2\left[c_{12}W_y + \frac{c_{22} - c_{11}}{2}W_x\right],$$

which in the zeroth approximation can be written in the matrix form

$$\left[\frac{1}{v_{s0}}\frac{\partial}{\partial t} + \frac{\partial}{\partial z}\right]\begin{Vmatrix} W_x \\ W_y \end{Vmatrix} = -iCP\begin{Vmatrix} W_x \\ W_y \end{Vmatrix}, \tag{2.2}$$

where

$$v_{s0} = \sqrt{\mu/\rho};\quad C = \frac{\omega(\mu + \nu_3)}{2v_{s0}\mu^2} = \frac{\pi(\mu + \nu_3)}{\lambda\mu^2};$$
$$P = \begin{Vmatrix} \dfrac{\sigma_{xx} - \sigma_{yy}}{2} & \sigma_{xy} \\ \sigma_{xy} & \dfrac{\sigma_{yy} - \sigma_{xx}}{2} \end{Vmatrix}.$$

In calculating the values of the acoustoelastic constant C we considered that

$$c_{11} - c_{22} = \frac{\mu + \nu_3}{\mu}(\sigma_{xx} - \sigma_{yy}),\ c_{12} = \frac{\mu + \nu_3}{\mu}\sigma_{xy}.$$

Earlier, Eqs. (2.2) had been derived only for the propagation of waves along the principal direction of the prestress tensor [5]. Naturally, the form of the equations in the lowest approximation does not depend on whether or not the direction of wave propagation coincides with the principal direction.

The quasitransverse waves are refined just as are the quasilongitudinal waves are: After finding W_x and W_y from (2.2), we obtain the longitudinal part of (1.5): $W_z = -(c_{13}W_x + c_{23}W_y)/(\lambda + \mu)$; the conservation law determines the amplitude and group velocity of the wave.

It is significant that the birefringence equations (2.2) coincide to within a constant C with the photoelasticity birefringence equations, whose solution was studied rather fully in [8]. The approximate solution of (2.2) relates the experimentally determined parameters and the stress in relatively simple form [12, 13]:

$$\Delta\cos 2\psi = C\int \sigma_{xx} + \sigma_{zz}dz,\ \Delta\sin 2\psi = 2C\int \sigma_{xz}dz. \tag{2.3}$$

Here the angle ψ (isocline parameter) and Δ (phase difference) of transverse waves are determined from measurements in the same way as in the case of a two-dimensional sample.

Using the equilibrium equations, we can transform the ray integrals (2.3) into the form [14, 15]

$$\begin{gathered}\int \sigma_{zz} dl = \frac{\partial}{\partial z} \int\limits_{m}^{m_1} H(m', \theta, z)\, dm' - A(m, \theta, z), \\ \int \frac{\partial}{\partial z} \sigma_{zz} dl = -\frac{\partial}{\partial m} H(m, \theta, z) + \sigma_{zz} \operatorname{ctg} \gamma \frac{d}{dm} \Gamma \Big|_{l_0}^{l_1},\end{gathered} \tag{2.4}$$

which reduces the problem of finding σ_{zz}, $\partial\sigma_{zz}/\partial z$ to the standard procedure of inverting the Radon transform at those points on condition that the contour is convex. In Eq. (2.4) m_1 is either of two extrme points of the projections of the contour of the cross section onto the m axis; γ is the angle between the z axis and n, the normal to the lateral surface; Γ is the arc length on the contour, measured from an arbitrary point; and the values of $\sigma_{zz}(\ell_1)$ and $\sigma_{zz}(\ell_0)$ at the ends of the ray are determined from the boundary conditions by tangential illumination at these points with the condition that the contour is convex.

Inclusion of the compatibility equations for the stresses makes it possible to determine the other components of the stress tensor of the solution of the first-order Lamé equations (not containing normal rotation) [15]. The specific algorithm for such reconstruction of an axisymmetric stressed state was considered in [16].

In acoustoelasticity the additional use of a longitudinal makes it possible to find the first invariant of the stress tensor. Indeed, the prestress-induced change that occurs in the transit time of the longitudinal wave (phase difference) is described by the ray integral

$$C_0 \int \sigma_{ll} + C_1 (\sigma_{mm} + \sigma_{zz})\, dl = C_0 K(m, \theta, z),$$

where C_0 and C_1 are constants determined by the parameters v_{p0}, α, and β of (1.8). The application of the inverse Radon transform to the linear combination of ray integrals

$$K + (1 - C_1) A = \int \sigma_{ll} + \sigma_{mm} + (1 + 2C_1)\, \sigma_{zz} dl$$

reproduces the value of the two-dimensional invariant $\sigma_{ll} + \sigma_{mm} = \sigma_2$; the value of σ_{zz} is assumed to be known from previous measurements (2.4). Using only equilibrium equations and the boundary conditions on the free lateral surface, we can thus determine the components σ_{xx} and σ_{yy} separately.

Indeed, when we eliminate σ_{xy}, σ_{xz}, and σ_{yz} from the equilibrium equations, we have

$$\frac{\partial^2}{\partial x^2} \sigma_{xx} - \frac{\partial^2}{\partial y^2} \sigma_{yy} = \frac{\partial^2}{\partial z^2} \sigma_{zz},$$

and by making the substitution $\sigma_{yy} = \sigma_2 - \sigma_{xx}$ we reduce it to the two-dimensional Poisson equation [15]

$$\Delta_{+} \sigma_{xx} = \frac{\partial^2}{\partial y^2} \sigma_2 + \frac{\partial^2}{\partial z^2} \sigma_{zz}$$

with the boundary condition $\sigma_{xx} = \sigma_2 n_y^2$. The value of σ_{xy} is found from the equilibrium equations by using the curvilinear integral of [15].

With the proposed simplest variant of the theory a generalization can be made quite easily to the case when nonlinear attenuation is included [17]. For stationary signals this generalization comes down to introducing third-order complex elastic constants. The methods of tomography of the tensor field of stresses with allowance for attenuation are modified in much the same way as is the tomography of scalar field [18] and so this will not be expounded here, especially since the experimental aspect of this topic has been studied very little thus far.

In conclusion, we note that when the difficulties involved in measuring the phase and polarization parameters of ultrasonic waves in three-dimensional objects are overcome, the results derived above make it possible to use tomographic methods for acoustodiagnostics of prestress [8, 15, 16].

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